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Scattering Theory Relevant to the Linear Transport of Particle Swarms

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The long-time behavior of the velocity distribution of a spatially uniform diluted guest population of charged particles moving within a host medium under the influence of a D.C. electric field is studied within the framework of scattering theory. We prove the existence of wave and scattering operators for a simplified one-dimensional model of the linearized Boltzmann equation. The theory is applied to the study of the long-term behavior of electrons and the occurrence of traveling waves in runaway processes.

KEY WORDS: Scattering theory; traveling waves; electron swarms.

1. INTRODUCTION

The physics of particle swarms has been studied extensively in the past decades and we refer to the recent report by Kumar[(1)] for an introduction to a phenomenological analysis of the subject matter and the experimental results (see also refs. 2 and 3). In a recent paper,[(4)] the time-dependent and the stationary problem for the linearized Boltzmann equation for charged particles under the influence of a spatially uniform D.C. electric field in a weakly ionized gas have been investigated. In spite of the theoretical understanding achieved in this area, some mathematical aspects pertaining to the long-term behavior of electrons require further study.

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The linearized equation describing the evolution of the space-averaged electron distribution $f(v, t)$ in a weakly ionized host medium as a function of the velocity $v \in \mathbb{R}$ and time $t \geqslant 0$ is as follows:

$$\frac{\partial f}{\partial t}(v, t) + a\frac{\partial f}{\partial v}(v, t) + \nu(v) f(v, t) = \int_{-\infty}^{+\infty} k(v, v')\, \nu(v')\, f(v', t)\, dv' \quad (1.1)$$

It is endowed with the initial condition

$$f(v, 0) = f_0(v) \quad (1.2)$$

The electrostatic acceleration a is assumed constant and positive. Recombination and ionization effects are assumed to balance each other so that the total number of charged particles is preserved. The expressions $\nu(v)$ and $k(v, v')$ denote the collision frequency (between an electron and the host medium) and the corresponding scattering kernel.

In ref. 4 the authors have proved the unique solvability of the time-dependent evolution system (1.1)–(1.2) in $L_1(\mathbb{R}, dv)$, as well as the nonnegativity of the solution for a nonnegative initial datum. They have also established necessary conditions and sufficient conditions for the existence of a (unique) nonnegative solution of the stationary problem. In this analysis, a crucial role is played by the dependence of the collision frequency $\nu(v)$ upon the speed v of the charged particles for large v. Under very minor assumptions on $\nu(v)$, a necessary condition for the stationary problem

$$a\frac{\partial f}{\partial v}(v) + \nu(v) f(v) = \int_{-\infty}^{+\infty} k(v, v')\, \nu(v')\, f(v')\, dv'$$

to admit a nontrivial nonnegative solution in $L_1(\mathbb{R}, dv)$ is that

$$\int_{-\infty}^{+\infty} \nu(v)\, dv = +\infty \quad (1.3)$$

In ref. 4 the relaxation of the solution to the stationary solution has been proved whenever it exists in $L_1(\mathbb{R}, dv)$. However, if instead of (1.3) we assume that

$$\int_{-\infty}^{+\infty} \nu(x)\, dv < +\infty \quad (1.4)$$

then the stationary solution does not exist in $L_1(\mathbb{R}, dv)$, but it does exist in $L_1(\mathbb{R}, \nu\, dv)$. In this case $\nu(v)$ decays too fast as $|v| \to \infty$ and collisions do not sufficiently slow down the most energetic charged particles as to enable relaxation to a nonzero steady state.[5] We may then expect the so-called runaway phenomenon. The purpose of this paper is to study the asymptotics of the problem under assumption (1.4). We prove that in this case the collisions generate a traveling wave in velocity space with "velocity" a.

In this paper scattering theory, a well-known tool for studying dynamical systems in many fields of physics, is used to investigate the large-time behavior of solutions to Eqs. (1.1)–(1.2). Suppose $W_0(t)$ is the group describing the free dynamics and $S(t)$ the semigroup describing the full dynamics of the electrons. Then wave operators are used to compare $S(t)\,g$ for the initial datum g (the full dynamics) to $W_0(t)h$ for a suitable initial datum h (the free dynamics). Such operators were introduced in the 1940s by physicists[6] and developed into a rigorous tool in the late 1950s.[7,8] Here we define the wave operators as linear maps on velocity space and follow the approach of Simon[9] for classical particles in a Hilbert space setting, subsequently adapted to the neutron transport equation by Hejtmanek[10] (see refs. 11–13 for further developments). For more general developments in mathematical scattering theory we refer to a number of monographs.[14-17]

Let us define the operators

$$\Omega^+ = \operatorname*{s\text{-}lim}_{t\to -\infty} S(-t)\, W_0(t), \qquad \Omega^- = \operatorname*{s\text{-}lim}_{t\to +\infty} W_0(-t)\, S(t) \tag{1.5}$$

as strong limits of operators on $L_1(\mathbb{R}, dv)$. For $s>0$ we multiply the first equation by $S(s)$ and the second equation by $W_0(s)$ to get the intertwining relation

$$S(s)\,\Omega^+ u = \operatorname*{s\text{-}lim}_{t\to -\infty} S(s-t)\, W_0(t-s)\, W_0(s)u = \Omega^+ W_0(s)u$$

i.e., Ω^+ connects the full dynamics with initial datum $\Omega^+ u$ to the free dynamics with initial datum u. A similar calculation gives

$$W_0(s)\Omega^- v = \operatorname*{s\text{-}lim}_{t\to +\infty} W_0(s-t)\, S(t-s)\, S(s)u = \Omega^- S(s)v$$

which allows for an analogous interpretation. Assuming, for the moment, the existence of Ω^+ and Ω^-, we define the scattering operator $\mathbb{S}$ as

$$\mathbb{S} = \Omega^- \Omega^+$$

Then we have for $s>0$

$$\mathbb{S}\, W_0(s) = \Omega^- \Omega^+ W_0(s) = \Omega^- S(s)\Omega^+ = W_0(s)\, \Omega^- \Omega^+ = W_0(s)\mathbb{S}$$

The equality of the leftmost and rightmost members will also be true for $s<0$ due to the group structure of $W_0(s)$. Thus the scattering operator

$\mathbb{S} = \Omega^- \Omega^+$ transforms a free solution which starts out as u^- near $t = -\infty$ into a free solution u^+ near $t = +\infty$.

The organization of the paper is as follows. In Section 2 we state the problem. In Section 3 we prove the existence of the so-called wave operators under the assumption of integrability of $\nu(v)$ on $\mathbb{R}$. In Section 4 we provide a proof of the existence of traveling waves in velocity space.

2. STATEMENT OF THE PROBLEM

In this section we give the basic notation and state some well-known results on the Cauchy problem (1.1)–(1.2). Prior to the functional formulation of the problem, let us introduce the Banach spaces $L_1(\mathbb{R}, dv)$ and $L_1(\mathbb{R}, \nu\, dv)$ with the norms $\|f\|_1 = \int_{-\infty}^{+\infty} |f(v)|\, dv$ and $\|f\|_\nu = \|\nu f\|_1$, respectively, and list the assumptions on a, $\nu(v)$, and $k(v, v')$.

Assumption (i): The acceleration a is a fixed positive constant;

Assumption (ii): The collision frequency $\nu(v)$ is a Lebesgue measurable, nonnegative, and even function of v on $\mathbb{R}$ which vanishes almost nowhere and is Lebesgue integrable on every bounded Lebesgue-measurable set.

Assumption (iii): The collision kernel $k(v, v') \geqslant 0$ appearing in the integral operator has the property

$$\int_{-\infty}^{+\infty} k(v, v')\, dv \equiv 1, \qquad v' \in \mathbb{R}$$

and, by reciprocity symmetry, we also have $k(-v, -v') = k(v, v')$.

We define $T_0 f = -a\, \partial f/\partial v$, $Af = -\nu(v) f$, and $(Kf)(v) = \int_{-\infty}^{+\infty} k(v, v')\, \nu(v') \times f(v')\, dv'$, where $D(T_0)$ is the set of those $f \in L_1(\mathbb{R}, dv)$ whose distributional derivative belongs to $L_1(\mathbb{R}, dv)$, $D(A)$ is the intersection of $L_1(\mathbb{R}, dv)$ and $L_1(\mathbb{R}, \nu\, dv)$, and K is a positive linear operator satisfying

$$\|Kf\|_1 = \|f\|_\nu, \qquad f \in L_1(\mathbb{R}, \nu\, dv) \quad \text{and} \quad f \geqslant 0$$

All our results will be true for abstract operators K: $L_1(\mathbb{R}, \nu\, dv) \to L_1(\mathbb{R}, dv)$ which are positive and satisfy $\|Kf\|_1 = \|f\|_\nu$ for all nonnegative $f \in L_1(\mathbb{R}, \nu\, dv)$. Using the preceding definitions, we can put problem (1.1)–(1.2) into the abstract form

$$\frac{df}{dt} = T_0 f(t) + Af(t) + Kf(t), \qquad t > 0$$

$$f(0) = f_0$$

where d/dt is the strong derivative, $f\colon \mathbb{R}^+ \to L_1(\mathbb{R}, dv)$, and f_0 is the initial datum.

Let us denote by $W_0(t)$ the strongly continuous evolution group

$$[W_0(t)\, g](v) = g(v - at), \qquad t \in \mathbb{R}$$

of isometries generated by the free streaming operator T_0, and by $S_0(t)$ the strongly continuous contraction semigroup

$$[S_0(t)\, g](v) = \exp\left[-\int_0^t \nu(v - as)\, ds\right] g(v - at) \tag{2.1}$$

generated by the streaming operator $T_0 - \nu$. Then $S_0(t)$ is a group of positive operators, which is bounded if and only if $\nu \in L_1(\mathbb{R}, dv)$. Further, in ref. 4 we have studied the operator $T = T_0 + A + K$ on the intersection $\mathscr{M}$ of $L_1(\mathbb{R}, dv)$, $L_1(\mathbb{R}, \nu\, dv)$, and the set of functions which are absolutely continuous on $[-b, b]$ for all $b > 0$, are of bounded variation and vanish at $-\infty$, and proved some closed extension of T to generate a strongly continuous semigroup $S(t)$ on $L_1(\mathbb{R}, dv)$ satisfying[4]

$$\|S(t) f\|_1 \leqslant \|f\|_1, \qquad f \geqslant 0 \quad \text{in} \quad L_1(\mathbb{R}, dv), \qquad t \geqslant 0$$

From now on we will not distinguish between the operator T with domain $\mathscr{M}$ and its closed extension in $L_1(\mathbb{R}, dv)$ generating $S(t)$. The full semigroup $S(t)$ cannot in general be extended to a group of positive operators. However, such an extension is possible if $\nu(v)$ is integrable, but also in many cases where $\nu(v)$ is not integrable [such as $\nu(v) \equiv \nu_0$ constant].

If we consider $S(t)$ as a perturbation of $S_0(t)$, we may derive the so-called Duhamel formulas. For their Laplace transforms

$$L_\lambda g = \int_0^\infty e^{-\lambda t} S_0(t)\, g\, dt, \qquad T_\lambda g = \int_0^\infty e^{-\lambda t} S(t)\, g\, dt, \qquad \operatorname{Re} \lambda > 0$$

we have the identities[4]

$$T_\lambda - L_\lambda = T_\lambda K L_\lambda, \qquad T_\lambda - L_\lambda = L_\lambda K T_\lambda, \qquad \operatorname{Re} \lambda > 0 \tag{2.2}$$

The first relation is true for every $\nu(v)$, but the latter only if T (with $\mathscr{M}$ as its domain) is closed in $L_1(\mathbb{R}, dv)$, which is the case if $\nu(v)$ is integrable on $\mathbb{R}$. In that case the operator L_λ can be defined for $\operatorname{Re} \lambda < 0$ as the resolvent of $T_0 + A$ [because $S_0(t)$ then is a bounded group]; we may then use the

[4] If the generator of $S(t)$ is the closure of T on $\mathscr{M}$, then $\|S(t) f\|_1 = \|f\|_1$, $f \geqslant 0$ in $L_1(\mathbb{R}, dv)$. This is true, e.g., if $\nu(v)$ is bounded or if (1.4) holds true.

positivity of L_λ for $\text{Re}\,\lambda < 0$ [due to the positivity of $S_0(t)$ for $t < 0$] to prove that, for $\text{Re}\,\lambda < 0$, KL_λ is bounded on $L_1(\mathbb{R}, dv)$ and $L_\lambda K$ is bounded on $L_1(\mathbb{R}, \nu\, dv)$, which allows us to generalize the relations (2.2) for $\text{Re}\,\lambda < 0$. For integrable $\nu(v)$ we then find the (bounded) group property of $S(t)$ as well as the Duhamel formulas

$$S(t) = S_0(t) + \int_0^t S(t-s)\, KS_0(s)\, ds, \qquad t \in \mathbb{R} \tag{2.3}$$

$$S(t) = S_0(t) + \int_0^t S_0(t-s)\, KS(s)\, ds, \qquad t \in \mathbb{R} \tag{2.4}$$

Again the former [i.e., (2.3)] is true for any $\nu(v)$, while the latter [i.e., (2.4)] is only true for integrable $\nu(v)$.

In ref. 4 we have established the following result.

Theorem 1. Suppose there exists a nontrivial solution φ of the stationary problem in $L_1(\mathbb{R}, dv)$. Then the semigroup $S(t)$ generated by T is mean ergodic, i.e., for every $g \in L_1(\mathbb{R}, dv)$ there exists a vector $Pg \in L_1(\mathbb{R}, dv)$ such that

$$\lim_{t \to \infty} \left\| \frac{1}{t} \int_0^t S(t')\, g\, dt' - Pg \right\|_1 = 0$$

The limit Pg is a one-dimensional projection of the form

$$(Pg)(v) = \alpha(g)\, \varphi(v), \qquad v \in \mathbb{R}$$

where $\alpha(g) = \int_{-\infty}^{\infty} \psi(v')\, g(v')\, dv'$ for some function $\psi \in L_\infty(\mathbb{R}, dv)$ with $\psi \geqslant 0$, $\int_{-\infty}^{+\infty} \psi(v')\, \varphi(v')\, dv' = 1$, and $\|\psi\|_\infty < +\infty$. If, in addition, the generator of $S(t)$ does not have purely imaginary eigenvalues, then

$$\lim_{t \to \infty} \|S(t)g - Pg\|_1 = 0, \qquad g \in L_1(\mathbb{R}, dv)$$

The assumption that T, the generator of $\{S(t)\}_{t \geqslant 0}$, does not have purely imaginary eigenvalues can be dropped in many cases, e.g., if $\nu(v)$ is bounded or if (1.4) holds true.[18,4]

3. THE EXISTENCE OF THE WAVE OPERATORS Ω^- AND Ω^+

In this section we prove the existence of the wave operators defined by (1.5) under the sufficient condition of integrability of $\nu(v)$ on $\mathbb{R}$. Condition (1.4) on the behavior of $\nu(v)$ as $|v| \to \infty$ will be sufficient to ensure the existence of Ω^-. We have the following result.

Theorem 2. If the collision frequency $\nu(v)$ satisfies Assumption (ii) and the additional assumption (1.4), then $\Omega^- = s\text{-}\lim_{t \to +\infty} W_0(-t)\, S(t)$ exists strongly in $L_1(\mathbb{R}, dv)$ and is a bounded positive operator.

Proof. Because $\int_0^t \nu(v-as)\, ds \leqslant (M/a)$ for all $t \in \mathbb{R}^+$, we have, for all positive $g \in L_1(\mathbb{R}, dv)$,

$$[S(t)\, g](v) \geqslant \exp\left[-\int_0^t \nu(v-as)\, ds\right] [W_0(t)\, g](v)$$
$$\geqslant \exp\left(-\frac{M}{a}\right) [W_0(t)\, g](v)$$

and consequently, for a.e. $v \in \mathbb{R}$ and $t \geqslant 0$, $g(v) \leqslant [S(t)g](v+at)\exp(M/a)$. Replacing g by $S(s)\, g$ and t by $t-s$ with $t \geqslant s \geqslant 0$, we obtain

$$[S(s)\, g](v) \leqslant [S(t)\, g][v + a(t-s)]\, \exp(M/a)$$

Hence,

$$\int_0^\infty \|S(s)g\|_\nu\, ds \leqslant \exp\left(\frac{M}{a}\right) \int_{-\infty}^\infty \frac{\nu(v)}{a} \|S(t)g\|_1\, dv = \frac{M}{a} \exp\left(\frac{M}{a}\right) \|g\|_1 < +\infty \tag{3.1}$$

We first compute

$$[W_0(-t)\, S_0(t)g](v) = [S_0(t)g](v+at) = \exp\left[-\int_0^t \nu(v + a(t-s))\, ds\right] g(v)$$

so that in the strong operator topology of $L_1(\mathbb{R}, dv)$

$$\Omega_0^- g = \lim_{t \to +\infty} W_0(-t)\, S_0(t)\, g, \qquad [\Omega_0^- g](v) = \exp\left[-\frac{1}{a}\int_v^\infty \nu(\hat{v})\, d\hat{v}\right] g(v)$$

which is a positive operator with a bounded positive inverse.

Next, we premultiply Eq. (2.4) by $S_0(-t)$ to obtain

$$S_0(-t)\, S(t)\, g = g + \int_0^t S_0(-s)\, KS(s)\, g\, ds$$

Using that the norm of $S_0(-s)$ on $L_1(\mathbb{R}, dv)$ is bounded above by $\exp(M/a)$, we find that the integral $\int_t^\infty \|S_0(-s)\, KS(s)\, g\|_1\, ds$ is finite [cf. (3.1)]. Hence, we have in the strong topology of $L_1(\mathbb{R}, dv)$

$$\Omega_1^- g = \lim_{t \to +\infty} S_0(-t)\, S(t)\, g = g + \int_0^\infty S_0(-s)\, KS(s)\, g\, ds$$

which is a bounded positive operator.

Finally, we see that Ω^- can be defined by $\Omega^- = \Omega_0^- \Omega_1^-$ and satisfies (1.5). ▮

For the existence of Ω^- no other condition on $\nu(v)$ apart from its integrability is necessary. This is reasonable from the physical point of view, because in our case $S(t)$ is an isometry for $t \geqslant 0$ and hence the number of charged particles is conserved. The same thing will appear to be valid for the existence of Ω^+.

Theorem 3. If the collision frequency $\nu(v)$ satisfies the same assumptions as in Theorem 1, then the limit $\Omega^+ = s\text{-}\lim_{t \to -\infty} S(-t)\, W_0(t)$ exists in the strong operator topology of $L_1(\mathbb{R}, dv)$ and is a bounded positive operator.

Proof. The proof is analogous to the proof of Theorem 2. ▮

4. TRAVELING WAVES IN THE ELECTRON TRANSPORT PROBLEM

When a nontrivial stationary solution exists in $L_1(\mathbb{R}, dv)$, the collision frequency $\nu(v)$ is not integrable.[4] A different phenomenon occurs if $\nu(v)$ is integrable. In this case there exists an (up to normalization) unique nontrivial nonnegative stationary solution in $L_1(\mathbb{R}, \nu\, dv)$ that does not belong to $L_1(\mathbb{R}, dv)$. Using the results of the preceding section, we prove that the solution of the time-dependent problem behaves as a traveling wave in velocity space with "velocity" a as $t \to +\infty$.

By a traveling wave (in velocity space) of "velocity" a we mean a function of the form $W_0(t)\, g$ with g independent of t, i.e., a function of the form

$$[W_0(t)\, g](v) = g(v - at)$$

In the study of the convergence of the solution of the time-dependent problem (1.1)–(1.2) to such a traveling wave an important role is played by the streaming semigroup $S_0(t)$. In the following lemma, we see how the integrability of $\nu(v)$ affects its asymptotic behavior.

Lemma 4. If $\nu(v)$ is integrable, we have

$$\lim_{t \to +\infty} \|S_0(t)\, g - W_0(t)\, \Omega_0^- g\|_1 = 0, \qquad g \in L_1(\mathbb{R}, dv) \tag{4.1}$$

where

$$(\Omega_0^- g)(v) = \exp\left[-\frac{1}{a} \int_v^{+\infty} \nu(v')\, dv' \right] g(v)$$

while for nonintegrable $\nu(v)$ we have $\lim_{t \to +\infty} \|S_0(t)\, g\|_1 = 0$, $g \in L_1(\mathbb{R}, dv)$.

Proof. Equation (4.1) follows directly by taking $t \to +\infty$ in the expression

$$\int_{-\infty}^{+\infty} \left\{ \exp\left[-\frac{1}{a} \int_{w}^{w+at} \nu(v'+at)\, dv' \right] - \exp\left[-\frac{1}{a} \int_{w}^{+\infty} \nu(v'+at)\, dv' \right] \right\} |g(w)|\, dw$$

for $\|S_0(t)\, g - W_0(t)\, \Omega_0^- g\|_1$, using dominated convergence. The result for nonintegrable $\nu(v)$ is an immediate consequence of definition (2.1) and the nonintegrability of $\nu(v)$. ∎

We have the following result.

Theorem 5. Let $\nu(v)$ be integrable. Then for every $g \in L_1(\mathbb{R}, dv)$, we have

$$\lim_{t \to +\infty} \|W_0(-t)\, S_0(t)\, g - \Omega_0^- g\|_1 = 0$$

$$\lim_{t \to +\infty} \|W_0(-t)\, S(t)\, g - \Omega^- g\|_1 = 0$$

where Ω_0^- and Ω^- are the wave operators defined in Section 2.

Proof. This theorem is a direct consequence of Lemma 4 and Theorem 2. ∎

Under the conditions of Theorem 5 we have

$$[S_0(t)\, g](v) \simeq [\Omega_0^- g](v - at), \qquad [S(t)\, g](v) \simeq [\Omega^- g](v - at)$$

as $t \to +\infty$. Hence, in the remote future both $S_0(t)\, g$ and $S(t)\, g$ behave like traveling waves in velocity space.

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